# MATH 6021 Lecture 8 11/2/2020

**Yau Conjecture (1982):** Given $(M^{n+1}, g)$ closed manifold, $\exists$ $\infty$'ly many closed embedded smooth min. hypersurfaces?

**Motivation:** Almgren-Pitts theory (codim 1 case) (embedded) & Sacks-Uhlenbeck theory (dim 2 surfaces) (immersed)

**$n=1$:** closed geodesics on closed surfaces.

**Birkhoff:** $(S^2, g)$ contains at least one non-trivial closed geodesic.

**Key Difficulties:**

(1) If $(S^2, g)$ has $K > 0$, then $\nexists$ stable closed geodesic.

(2) $\pi_1(S^2) = 0 \Rightarrow \nexists$ non-contractible loops

$\Rightarrow$ "minimization" approach DOES NOT WORK!

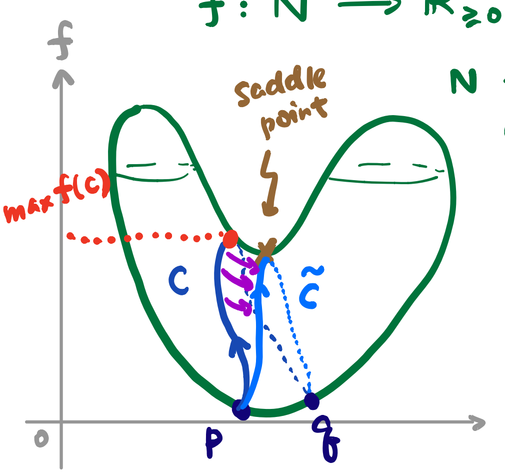
**Idea:** Need a new way to produce unstable critical points

$\rightsquigarrow$ "min-max theory"

**Main Idea:** "Mountain-pass Lemma"

$f : N \to \mathbb{R}_{\geq 0}$

N finite dim. manifold




Suppose $f(p) = f(q) = 0 = \min_N f$

Consider path $c : [0,1] \to N$ s.t.

- $c(0) = p$ ; $c(1) = q$
- $c$ is "homotopically non-trivial"

**Define:** $W := \inf_c \left( \sup_{s \in [0,1]} f(c(s)) \right) > 0$

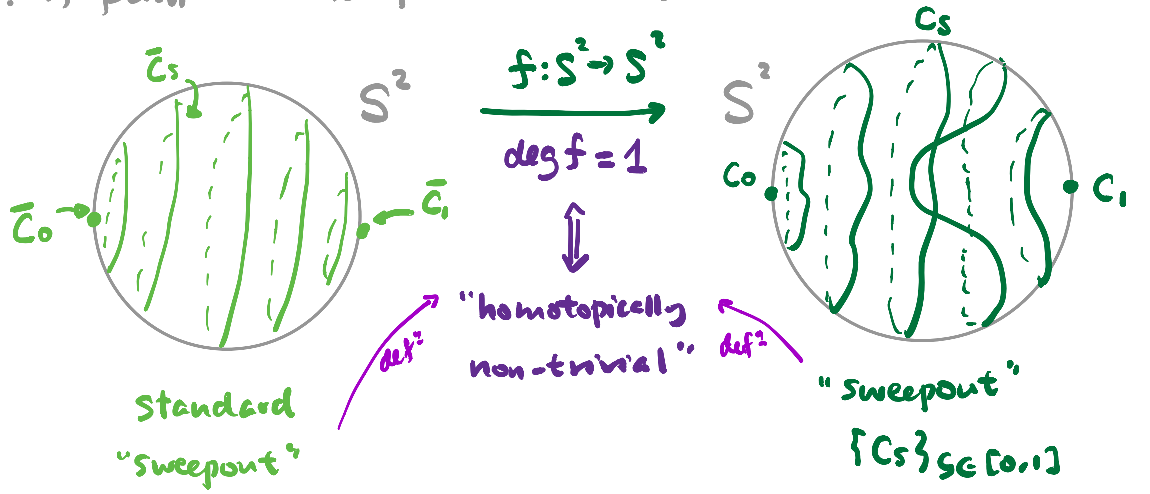
is achieved by some "optimal" mountain-pass $\tilde{c}$.

In our case, to find closed geodesics on $(S^2, g)$, we consider

Length functional $\quad \mathcal{L} : \left\{ \begin{array}{c} \text{closed loops} \\ \text{in } S^2 \end{array} \right\} \longrightarrow \mathbb{R}_{\geq 0}$

$\uparrow$ $\infty$-dim $\mathcal{N}$

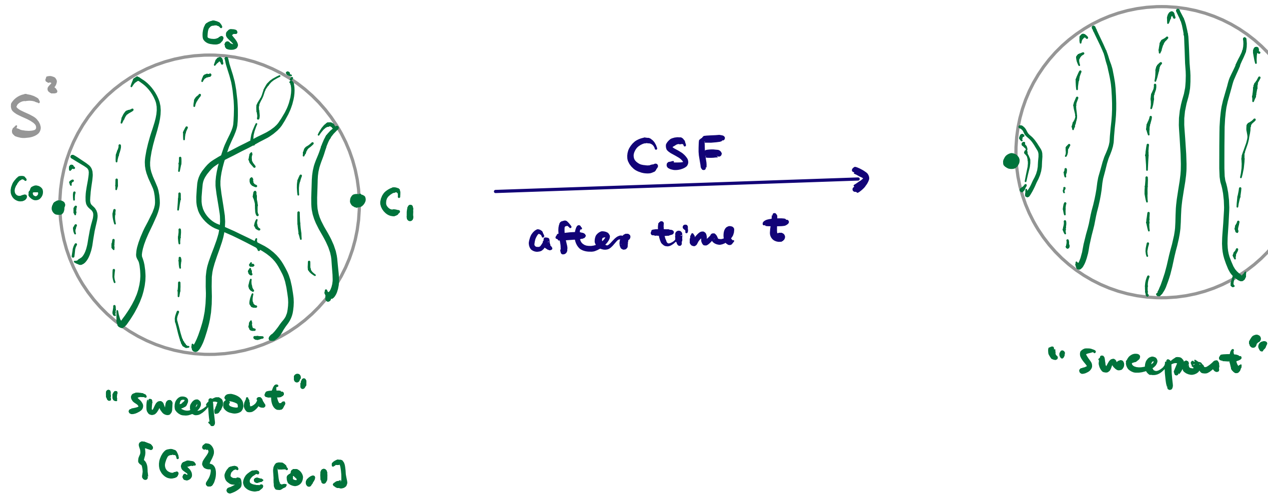
Remark: A "path" in this space is a 1-parameter family of closed loops.




**Thm** (Birkhoff 1917) $\exists$ (non-trivial) closed geodesic $\gamma$ in $(S^2, g)$ s.t.

$$\mathcal{L}(\gamma) = W := \inf_{\{C_s\}} \left( \sup_{s \in [0,1]} \mathcal{L}(C_s) \right) > 0$$

Idea of Proof: Apply the "Curve Shortening Flow" (CSF) simultaneously to all the curves $\{C_s\}_{s \in [0,1]}$ in a sweepout:




Grayson 1989: For (CSF) of embedded loops on a closed surface,

either (i) loop shrinks to a point in finite time

or (ii) loop converges to a closed geodesic as $t \to +\infty$.

$\deg(f) = 1 \Rightarrow$ Some $C_s$ does NOT shrink to a point.

By Grayson, $C_s \to$ geodesic (non-trivial). $\square$

**Remark:** Birkhoff did not have (CSF) available at that time, he used an ad-hoc "discrete" curve shortening process.

**Q:** What about in higher dimensions?

New Difficulties
- regularity of min. surfaces
- mean curvature flow develop singularities.

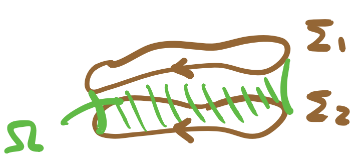
## Almgren-Pitts Min-Max Theory

**Q:** How to produce variationally unstable k-dim. minimal submanifolds in $(M^{n+1}, g)$?

**Recall:** ( Integral currents in $M^{n+1} \xrightarrow{\text{Nash}} \mathbb{R}^N$ )

**Def$^n$:** $\mathcal{Z}_k(M;\mathbb{Z}) :=$ the space of k-dim'l integral cycles in $M$ with the "flat" topology

**Recall:** $\Sigma_1, \Sigma_2 \in \mathcal{Z}_k(M;\mathbb{Z})$ are "close" in flat topology if $\exists$ (k+1)-dim'l integral current $\Omega$ in $M$ s.t. $\partial\Omega = \Sigma_1 - \Sigma_2$ & $\mathrm{Vol}(\Omega)$ is "small"




**Fact:** flat topology $\Leftrightarrow$ weak topology

**Q:** What is the "topology" of $\mathcal{Z}_k(M;\mathbb{Z})$?

Almgren computed all the homotopy groups of $\mathcal{Z}_k(M;\mathbb{Z})$.

## Almgren Isomorphism (1962)

$$\pi_\ell\left(\mathcal{Z}_k(M;\mathbb{Z})\right) \cong H_{k+\ell}(M;\mathbb{Z}) \qquad \text{for all } \ell, k$$

In particular,

- $k=\ell=1$, $M=S^2$ $\Rightarrow$ $\pi_1(\mathcal{Z}_1(S^2;\mathbb{Z})) \cong H_2(S^2;\mathbb{Z}) \cong \mathbb{Z}$.
- $k=n$, $M^{n+1}$ orientable $\Rightarrow$ $\pi_1\left(\mathcal{Z}_n(M;\mathbb{Z})\right) \overset{(*)}{\cong} H_{n+1}(M^{n+1};\mathbb{Z}) \cong \mathbb{Z}$.

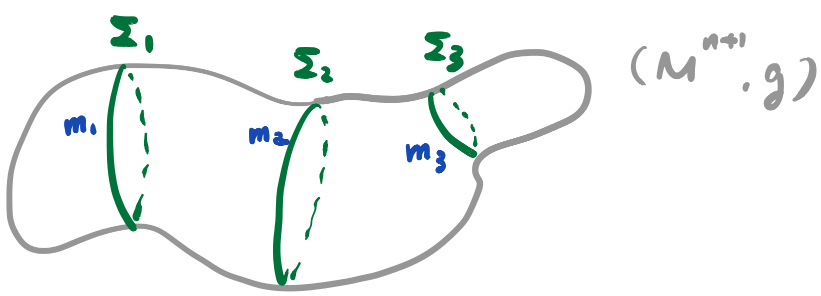
**Def$^{\underline{n}}$:** A "sweepout" of $M$ is a 1-parameter family $\{\Sigma_t\}_{t\in[0,1]}$ of integral $n$-cycles s.t. $\Sigma_0, \Sigma_1 = \text{pt.}$ ($=0$ in $\mathcal{Z}_n(M;\mathbb{Z})$) and $\{\Sigma_t\}_{t\in[0,1]} \in \pi_1(\mathcal{Z}_n(M;\mathbb{Z}))$ represents $[M] \in H_{n+1}(M;\mathbb{Z})$ under $(*)$. Define:

$$W := \inf_{\substack{\{\Sigma_t\} \\ \text{sweepout}}} \left( \sup_{t\in[0,1]} \overbrace{\text{Area}(\Sigma_t)}^{\mathbb{M}} \right) \qquad \text{"width"}$$

## Min-Max Theorem: (Almgren '65, Pitts '81, Schoen-Simon '81)

Suppose $3 \le n+1 \le 7$. Then, $\exists$ disjoint collection of smooth, closed, embedded min. hypersurfaces $\{\Sigma_1, \dots, \Sigma_g\}$ of $M^{n+1}$ s.t.

$$W = m_1 \text{Area}(\Sigma_1) + \dots + m_g \text{Area}(\Sigma_g) \quad \text{for some } m_i \in \mathbb{N}.$$



Remarks : (i) isoperimetric ineq. $\Rightarrow$ $W > 0$

$\Rightarrow$ existence of ONE nontrivial min. hypersurface.

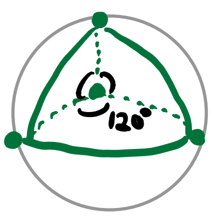
(ii) Almgren proved only the "existence" in any dimensions & codimensions, in the sense "stationary integral varifolds".

(iii) Regularity in codim 1 case by Pitts ($n+1 \leq 6$), Schoen-Simon ($n+1 = 7$).

(iv) For $n+1 \geq 8$, the same result holds except that $\Sigma_i$'s may have a singular set of codim $\geq 7$.

[cf. related to regularity of stable min. hypersurface, Schoen-Simon '81 ]

(v) For $n+1 = 2$, the Almgren-Pitts theory may not produce closed geodesics but stationary geodesic networks.

( E.g.  )



The proof of Min-Max Thm consists of 2 parts:

(I) Existence :

$\exists$ non-trivial "weakly defined" hypersurface which is "minimal" in the sense of varifolds.

(II) Regularity :

the min-max varifold enjoys nice variational property called "almost minimizing property" which shares similar regularity properties with stable min. hypersurfaces.

# Some Remarks :

(i) Recall: $\mathbb{M}$ of currents is ONLY lower semi-continuous, i.e.

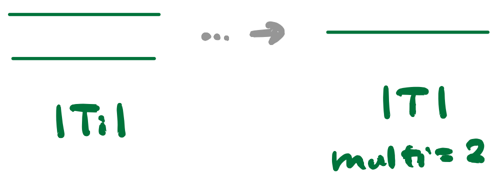
$$\underset{\text{as currents}}{T_i \rightharpoonup T} \quad \Rightarrow \quad \mathbb{M}(T) \leq \liminf_{i \to \infty} \mathbb{M}(T_i)$$

O.K. for minimization

Not O.K. for min-max construction

mass cancellation may occur :


Solution: consider "varifolds" in the measure theoretic sense




Area/mass for varifolds are continuous functional (w.r.t. some weak topology)

(ii) Heuristically,

The critical pt $\gamma$ has index $= 1$.

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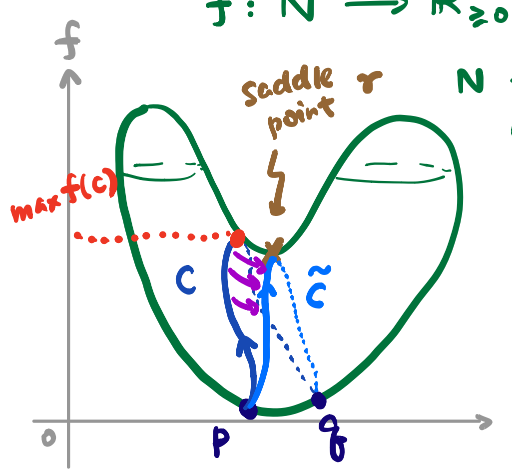
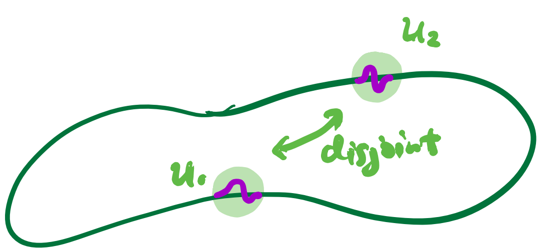
expect the min-max min. hypersurface constructed have index $= 1$.

$f : N \to \mathbb{R}_{\geq 0}$

N finite dim. manifold




$\Sigma$ index $= 1$

⇓

Stable in either $\mathcal{U}_1$, $\mathcal{U}_2$

( Ref: Colding - De Lellis 2003, in Surveys in Diff. Geom. )